Programming chemistry in DNA addressable bioreactors
Supplementary information

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Appendix A: Grammar and equivalence relations

States:

$$\mathsf{S} := \emptyset \mid \mathsf{S} \circ \mathsf{S} \mid x_i : \mathsf{P} \tag{1}$$
$$\mathsf{P} := 0 \mid \mathsf{P} + \mathsf{P} \mid \mathsf{q}^* (\!(\mathsf{P})\!) \mid \mathsf{q} \mid m_j \tag{2}$$
$$\mathsf{q} := \mathsf{s} \mid \mathsf{s}^* \rhd \mathsf{s}^* \tag{3}$$

$$S \circ (S' \circ S'') \equiv (S \circ S') \circ S'' \tag{6}$$
$$S \circ S' \equiv S' \circ S \tag{7}$$
$$S \circ \emptyset \equiv S \tag{8}$$
$$P + (P' + P'') \equiv (P + P') + P'' \tag{9}$$
$$P + P' \equiv P' + P \tag{10}$$
$$P + 0 \equiv P \tag{11}$$
$$q_1 + (q_2 + q_3) \equiv (q_1 + q_2) + q_3 \tag{12}$$
$$q_1 + q_2 \equiv q_2 + q_1 \tag{13}$$
$$q + \diamond \equiv q \tag{14}$$
$$s_1 + (s_2 + s_3) \equiv (s_1 + s_2) + s_3 \tag{15}$$
$$s_1 + s_2 \equiv s_2 + s_1 \tag{16}$$
$$s + \diamond \equiv s \tag{17}$$
$$x_i : P \circ x_i : P' \equiv x_i : P + P' \tag{18}$$

$$\frac{S_1 \equiv S_2}{S \circ S_1 \equiv S \circ S_2} \tag{19}$$
$$\frac{P_1 \equiv P_2}{x_i : P_1 \equiv x_i : P_2} \tag{20}$$
$$\frac{P_1 \equiv P_2}{P + P_1 \equiv P + P_2} \tag{21}$$
$$\frac{q_1 \equiv q_2}{q^* + q_1 \equiv q^* + q_2} \tag{22}$$
$$\frac{s_1 \equiv s_2}{s^* + s_1 \equiv s^* + s_2} \tag{23}$$
$$\frac{s_1^* \equiv s_2^*}{s_1^* \rhd s^* \equiv s_2^* \rhd s^*} \tag{24}$$
$$\frac{s_1^* \equiv s_2^*}{s^* \rhd s_1^* \equiv s^* \rhd s_2^*} \tag{25}$$
$$\frac{P_1 \equiv P_2}{q^* (\!(P_1)\!) \equiv q^* (\!(P_2)\!)} \tag{26}$$
$$\frac{q_1^* \equiv q_2^*}{q_1^* (\!(P)\!) \equiv q_2^* (\!(P)\!)} \tag{27}$$

Transitions:

$$s_1^* \rhd s_2^* \; + \; s_1^* \longrightarrow s_2^* \tag{38}$$
$$\frac{s + s' \longrightarrow s''}{s + (s' + t)(\!(P)\!) \longrightarrow (s'' + t)(\!(P)\!)} \tag{39}$$
$$\frac{q + q' \longrightarrow q''}{(q + q')(\!(P)\!) \longrightarrow q''(\!(P)\!)} \tag{40}$$

$$\frac{P \longrightarrow P'}{P + P'' \longrightarrow P' + P''} \tag{32}$$
$$\frac{P \longrightarrow P'}{q^* (\!(P)\!) \longrightarrow q^* (\!(P')\!)} \tag{33}$$
$$\frac{P \longrightarrow P'}{x_i : P \longrightarrow x_i : P'} \tag{34}$$
$$\frac{S \longrightarrow S'}{S \circ S'' \longrightarrow S' \circ S''} \tag{35}$$
$$\frac{I : S \longrightarrow S'}{I : S \circ \bar{S} \longrightarrow S' \circ \bar{S}} \tag{50}$$
$$\frac{P \equiv P' \qquad P' \longrightarrow P'' \qquad P'' \equiv P'''}{P \longrightarrow P'''} \tag{36}$$
$$\frac{S \equiv S' \qquad S' \longrightarrow S'' \qquad S'' \equiv S'''}{S \longrightarrow S'''} \, . \tag{37}$$

$$\emptyset \xrightarrow{\textbf{feed}(x,m_i,\nu)} x : (\!(\nu m_i)\!) \tag{41}$$
$$\emptyset \xrightarrow{\textbf{feed\_tag}(x,s,\nu)} x : \nu s \tag{42}$$
$$x : (s + q^*)(\!(P)\!) \xrightarrow{\textbf{move}(s,x,z)} z : (s + q^*)(\!(P)\!) \tag{43}$$
$$x : s \xrightarrow{\textbf{move}(s,x,z)} z : s \tag{44}$$
$$x : s + q^*(\!(P)\!) \xrightarrow{\textbf{tag}(x)} x : (s + q^*)(\!(P)\!) \tag{45}$$
$$x : q_1^*(\!(P)\!) + q_2^*(\!(P')\!) \xrightarrow{\textbf{fuse}(x)} x : (q_1^* + q_2^*)(\!(P + P')\!) \tag{46}$$
$$x : P \xrightarrow{\textbf{flush}(x)} \emptyset \tag{47}$$
$$x : P \xrightarrow{\textbf{wrap}(x,z)} z : (\!(P)\!) \tag{48}$$
$$x : q^*(\!(P)\!) \xrightarrow{\textbf{burst}(x)} x : q^*(\!(\;)\!) + P \tag{49}$$

Programs:

$$\pi := \epsilon \;\mid\; I;\pi \tag{51}$$

$$\epsilon\mathbf{;}\pi = \pi \tag{52}$$

$$(I;\pi)\mathbf{;}\pi' = I;(\pi\mathbf{;}\pi') \tag{53}$$

$$\langle \epsilon, S\rangle \longrightarrow S \tag{54}$$

$$\frac{\langle \pi, S''\rangle \longrightarrow S \qquad I : S' \longrightarrow S''}{\langle I;\pi, S'\rangle \longrightarrow S} \tag{55}$$

$$\frac{\langle \pi, S'\rangle \longrightarrow S}{\langle I;\pi, S'\rangle \longrightarrow S} \tag{56}$$

$$\frac{S \longrightarrow S' \qquad \langle \pi, S'\rangle \longrightarrow S''}{\langle \pi, S\rangle \longrightarrow S''} \tag{57}$$

$$\frac{\langle \pi, S\rangle \longrightarrow S' \qquad S' \longrightarrow S''}{\langle \pi, S\rangle \longrightarrow S''} \tag{58}$$

$$\frac{S \equiv S'' \qquad \langle \pi, S\rangle \longrightarrow S' \qquad S' \equiv S'''}{\langle \pi, S''\rangle \longrightarrow S'''} \tag{59}$$

Appendix B: Proofs

Proof of Lemma 1

We prove the statement:

$$\langle \pi, S\rangle \longrightarrow S' \quad \Longrightarrow \quad \left\langle \pi, S \circ \bar{S}\right\rangle \longrightarrow S' \circ \bar{S}. \tag{60}$$

Given that there exists an inference tree that derives the clause $\langle \pi, S\rangle \longrightarrow S'$, we show that there exists a parallel inference tree that derives $\left\langle \pi, S \circ \bar{S}\right\rangle \longrightarrow S' \circ \bar{S}$. The statement is proven by induction over the structure of these derivation.

Proof:

A. Assume that $\langle \pi, S\rangle \longrightarrow S'$ was derived by (54) in the last step. Hence, $\pi = \epsilon$ and $S' = S$. It trivially follows that $\left\langle \epsilon, S \circ \bar{S}\right\rangle \longrightarrow S \circ \bar{S}$, as axiom (54) holds for any state.

B. Assume that $\langle \pi, S\rangle \longrightarrow S'$ was derived by (55) in the last step. Hence, $\pi = I';\pi'$.

$$\frac{\langle \pi', S''\rangle \longrightarrow S' \qquad I : S \longrightarrow S''}{\langle I;\pi', S\rangle \longrightarrow S'} \tag{55}$$

We have as induction hypothesis that $\langle \pi', S''\rangle \longrightarrow S' \Longrightarrow \left\langle \pi', S'' \circ \bar{S}\right\rangle \longrightarrow S' \circ \bar{S}$. We can then use rule (55) to derive:

$$\frac{\left\langle \pi', S'' \circ \bar{S}\right\rangle \longrightarrow S' \circ \bar{S} \qquad \dfrac{I : S \longrightarrow S''}{I : S \circ \bar{S} \longrightarrow S'' \circ \bar{S}}\,(50)}{\left\langle I;\pi', S \circ \bar{S}\right\rangle \longrightarrow S' \circ \bar{S}}\,(55)$$

Hence, if $\left\langle \pi', S'' \circ \bar{S}\right\rangle \longrightarrow S' \circ \bar{S}$ is derivable, $\left\langle I;\pi', S \circ \bar{S}\right\rangle \longrightarrow S' \circ \bar{S}$ is derivable by induction.

C. Assume that $\langle \pi, S\rangle \longrightarrow S'$ was derived by (56) in the last step. Hence, $\pi = I;\pi'$.

$$\frac{\langle \pi', S\rangle \longrightarrow S'}{\langle I;\pi', S\rangle \longrightarrow S'} \tag{56}$$

We have as induction hypothesis that $\langle \pi', S\rangle \longrightarrow S' \Longrightarrow \left\langle \pi', S \circ \bar{S}\right\rangle \longrightarrow S' \circ \bar{S}$. Rule (56) holds for any state, thus we can derive, with S substituted by $S \circ \bar{S}$:

$$\frac{\left\langle \pi', S \circ \bar{S}\right\rangle \longrightarrow S' \circ \bar{S}}{\left\langle I;\pi', S \circ \bar{S}\right\rangle \longrightarrow S' \circ \bar{S}}\,(56)$$

Hence, if $\left\langle \pi', S \circ \bar{S}\right\rangle \longrightarrow S' \circ \bar{S}$ is derivable, $\left\langle I;\pi', S \circ \bar{S}\right\rangle \longrightarrow S' \circ \bar{S}$ is derivable by induction.

D. Assume that $\langle \pi, S \rangle \longrightarrow S'$ was derived by (57) in the last step.

$$\frac{S \longrightarrow S'' \qquad \langle \pi, S'' \rangle \longrightarrow S'}{\langle \pi, S \rangle \longrightarrow S'} \tag{57}$$

We have as induction hypothesis that $\langle \pi, S'' \rangle \longrightarrow S' \Longrightarrow \left\langle \pi, S'' \circ \bar{S} \right\rangle \longrightarrow S' \circ \bar{S}$. We can then use rule (35) to derive:

$$\frac{\dfrac{S \longrightarrow S''}{S \circ \bar{S} \longrightarrow S'' \circ \bar{S}}\,(35) \qquad \left\langle \pi, S'' \circ \bar{S} \right\rangle \longrightarrow S' \circ \bar{S}}{\left\langle \pi, S \circ \bar{S} \right\rangle \longrightarrow S' \circ \bar{S}}\,(57)$$

Hence, if $\left\langle \pi, S'' \circ \bar{S} \right\rangle \longrightarrow S' \circ \bar{S}$ is derivable, $\left\langle \pi, S \circ \bar{S} \right\rangle \longrightarrow S' \circ \bar{S}$ is derivable by induction.

E. Assume that $\langle \pi, S \rangle \longrightarrow S'$ was derived by (58) in the last step.

$$\frac{\langle \pi, S \rangle \longrightarrow S'' \qquad S'' \longrightarrow S'}{\langle \pi, S \rangle \longrightarrow S'} \tag{58}$$

We have as induction hypothesis that $\langle \pi, S \rangle \longrightarrow S'' \Longrightarrow \left\langle \pi, S \circ \bar{S} \right\rangle \longrightarrow S'' \circ \bar{S}$. Again, we can use rule (35) to derive:

$$\frac{\left\langle \pi, S \circ \bar{S} \right\rangle \longrightarrow S'' \circ \bar{S} \qquad \dfrac{S'' \longrightarrow S'}{S'' \circ \bar{S} \longrightarrow S' \circ \bar{S}}\,(35)}{\left\langle \pi, S \circ \bar{S} \right\rangle \longrightarrow S' \circ \bar{S}}\,(58)$$

Hence, if $\left\langle \pi, S \circ \bar{S} \right\rangle \longrightarrow S'' \circ \bar{S}$ is derivable, $\left\langle \pi, S \circ \bar{S} \right\rangle \longrightarrow S' \circ \bar{S}$ is derivable by induction.

F. Finally, assume that $\langle \pi, S \rangle \longrightarrow S'$ was derived by (59) in the last step.

$$\frac{S \equiv S'' \qquad \langle \pi, S'' \rangle \longrightarrow S''' \qquad S''' \equiv S'}{\langle \pi, S \rangle \longrightarrow S'} \tag{59}$$

We have as induction hypothesis that $\langle \pi, S'' \rangle \longrightarrow S''' \Longrightarrow \left\langle \pi, S'' \circ \bar{S} \right\rangle \longrightarrow S''' \circ \bar{S}$. Thus, we can infer:

$$\frac{\dfrac{S \equiv S''}{S \circ \bar{S} \equiv S'' \circ \bar{S}}\,(19) \qquad \left\langle \pi, S'' \circ \bar{S} \right\rangle \longrightarrow S''' \circ \bar{S} \qquad \dfrac{S''' \equiv S'}{S''' \circ \bar{S} \equiv S' \circ \bar{S}}\,(19)}{\left\langle \pi, S \circ \bar{S} \right\rangle \longrightarrow S' \circ \bar{S}}\,(59)$$

Hence, if $\left\langle \pi, S'' \circ \bar{S} \right\rangle \longrightarrow S''' \circ \bar{S}$ is derivable, then $\left\langle \pi, S \circ \bar{S} \right\rangle \longrightarrow S' \circ \bar{S}$ is derivable by induction.

Proof of Lemma 2

We prove the statement:

$$\left\langle \pi', S \right\rangle \longrightarrow S' \quad \wedge \quad \left\langle \pi'', S' \right\rangle \longrightarrow S'' \quad \Longrightarrow \quad \left\langle \pi'; \pi'', S \right\rangle \longrightarrow S'' \tag{61}$$

Proof: Again, the proof is performed inductively over the structure of the derivation, in particular the derivation of the clause $\langle \pi', S \rangle \longrightarrow S'$ for arbitrary derivations of the clause $\langle \pi'', S' \rangle \longrightarrow S''$.

A. Assume that $\langle \pi', S \rangle \longrightarrow S'$ was derived by (54) in the last step. Hence, $\pi' = \epsilon$ and $S' = S$.

$$\langle \epsilon, S \rangle \longrightarrow S \tag{54}$$

For any π'' with $\langle \pi'', S' \rangle \longrightarrow S''$, we then have $\langle \pi'', S' \rangle \longrightarrow S'' = \langle \epsilon; \pi'', S' \rangle \longrightarrow S''$ because $\epsilon; \pi'' = \pi''$ by definition of ';' among programs.

B. Assume that $\langle \pi', S\rangle \longrightarrow S'$ was derived by (55) in the last step. Hence, $\pi' = I;\pi'''$. As induction hypothesis we have that $\langle \pi''';\pi'', S'\rangle \longrightarrow S''$ is derivable for any π''. We can then infer that

$$\frac{\langle \pi''';\pi'', S'\rangle \longrightarrow S'' \qquad I : S \longrightarrow S'}{\langle I;\pi''';\pi'', S\rangle \longrightarrow S''}\ (55)$$

Hence, if $\langle \pi''';\pi'', S'\rangle \longrightarrow S''$ is derivable, we know by induction that $\langle \pi';\pi'', S\rangle \longrightarrow S''$ is derivable.

C. Assume that $\langle \pi', S\rangle \longrightarrow S'$ was derived by (56) in the last step. Hence, $\pi' = I;\pi'''$. We have as induction hypothesis that $\langle \pi''';\pi'', S'\rangle \longrightarrow S''$ is derivable for any π''. We can then infer that

$$\frac{\langle \pi''';\pi'', S\rangle \longrightarrow S''}{\langle I;\pi''';\pi'', S\rangle \longrightarrow S''}\ (56)$$

Hence, if $\langle \pi''';\pi'', S\rangle \longrightarrow S''$ is derivable, we know by induction that $\langle \pi';\pi'', S\rangle \longrightarrow S''$ is derivable.

D. Assume that $\langle \pi', S\rangle \longrightarrow S'$ was derived by (57) in the last step. As induction hypothesis we have that $\langle \pi';\pi'', S'''\rangle \longrightarrow S''$ is derivable for any π''. We can then infer that

$$\frac{S \longrightarrow S''' \qquad \langle \pi';\pi'', S'''\rangle \longrightarrow S''}{\langle \pi';\pi'', S\rangle \longrightarrow S''}\ (57)$$

Hence, if $\langle \pi';\pi'', S'''\rangle \longrightarrow S''$ is derivable, we know by induction that $\langle \pi';\pi'', S\rangle \longrightarrow S''$ is derivable.

E. Assume that $\langle \pi', S\rangle \longrightarrow S'$ has been derived by (58) in the last step:

$$\frac{\langle \pi', S\rangle \longrightarrow \bar{S} \qquad \bar{S} \longrightarrow S'}{\langle \pi', S\rangle \longrightarrow S'} \tag{58}$$

We have the second hypothesis $\langle \pi'', S'\rangle \longrightarrow S''$, and we need to show that $\langle \pi';\pi'', S\rangle \longrightarrow S''$. By second hypothesis and (57) with $\bar{S} \longrightarrow S'$ we obtain:

$$\frac{\bar{S} \longrightarrow S' \qquad \langle \pi'', S'\rangle \longrightarrow S''}{\langle \pi'', \bar{S}\rangle \longrightarrow S''}\ (57)$$

Hence, if we have as induction hypothesis that $\langle \pi', S\rangle \longrightarrow \bar{S}$ and $\langle \pi'', \bar{S}\rangle \longrightarrow S''$ then we can follow $\langle \pi';\pi", S\rangle \longrightarrow S''$.

F. Finally, assume that $\langle \pi', S\rangle \longrightarrow S'$ was derived by (59) in the last step:

$$\frac{S \equiv S_0 \qquad \langle \pi', S_0\rangle \longrightarrow S_1 \qquad S_1 \equiv S'}{\langle \pi', S\rangle \longrightarrow S'} \tag{59}$$

We are in the case $\langle \pi', S_0\rangle \longrightarrow S_1$ with $S \equiv S_0$ and $S_1 \equiv S'$. Our second hypothesis is that $\langle \pi'', S'\rangle \longrightarrow S''$, and we need to show that $\langle \pi';\pi'', S\rangle \longrightarrow S''$. By (59) we then have also that $\langle \pi'', S_1\rangle \longrightarrow S''$:

$$\frac{S_1 \equiv S' \qquad \langle \pi'', S'\rangle \longrightarrow S''}{\langle \pi'', S_1\rangle \longrightarrow S''}\ (59)$$

By induction hypothesis we have that $\langle \pi', S_0\rangle \longrightarrow S_1$ and $\langle \pi'', S_1\rangle \longrightarrow S'' \Longrightarrow \langle \pi';\pi'', S_0\rangle \longrightarrow S''$. By (59) again, $\langle \pi';\pi'', S\rangle \longrightarrow S''$.

Proof of Theorem 3

Proof: by induction over the structure of S.

A. Case $S \equiv \emptyset$.
This holds in the initial state. The program to create $S \equiv \emptyset$ is ϵ: $\langle \epsilon, \emptyset \rangle \rightarrow \emptyset$.

B. Case $S \equiv S' \circ S''$ with $S', S'' \in \omega^{+}(\Pi_{\mathbf{min}})$.
By induction, there exist programs π' and π'' such that $\langle \pi', \emptyset \rangle \longrightarrow S'$ and $\langle \pi'', \emptyset \rangle \longrightarrow S''$. Lemma 1 then implies that $\langle \pi', \emptyset \circ \emptyset \rangle \longrightarrow S' \circ \emptyset$, and $\langle \pi'', S' \circ \emptyset \rangle \longrightarrow S' \circ S''$. It follows with lemma 2 that $\langle \pi'; \pi'', \emptyset \circ \emptyset \rangle \longrightarrow S' \circ S''$. But $\emptyset \circ \emptyset \equiv \emptyset$ and $S' \circ S'' \equiv S$, hence $\langle \pi'; \pi'', \emptyset \rangle \longrightarrow S$ due to inference rule (59).

C. Case $S \equiv x_i : P$.

C.1. Subcase $P \equiv 0$.
This holds in the initial state. The program to create $S \equiv x_i : 0$ is ϵ: $\langle \epsilon, \emptyset \rangle \rightarrow x_i : 0$.

C.2. Subcase $P \equiv P' + P''$.
This case can be reduced to case B. by means of the distributive relation (18): $x_i : P' + P'' \equiv x_i : P' \circ x_i : P''$.

C.3. Subcase $P \equiv q \llparenthesis\, P' \,\rrparenthesis$ with $d(P') = 0$.

C.3.1. Subsubcase $P \equiv \llparenthesis\, 0 \,\rrparenthesis$.
This is achieved by the program $\pi = \mathbf{feed}(x_j, m, 0); \mathbf{transport}(x_j, x_i, x_k, \sigma)$ for arbitrary $m \in \mathcal{M}$, $x_j \in \mathcal{L}_{\mathbf{feed}}$, $x_k \in \mathcal{L}_{\mathbf{feed\_tag}}$, $\sigma \in \mathcal{T}$.

C.3.2. Subsubcase $P \equiv \llparenthesis\, m_j \,\rrparenthesis$.
This is achieved by the program $\pi = \mathbf{feed}(x_j, m_j, 1); \mathbf{transport}(x_j, x_i, x_k, \sigma)$ for $x_j \in \mathcal{L}_{\mathbf{feed}}$, $x_k \in \mathcal{L}_{\mathbf{feed\_tag}}$, $\sigma \in \mathcal{T}$.

C.3.3. Subsubcase $P \equiv \llparenthesis\, s_k \,\rrparenthesis$.
Since $d(\llparenthesis\, s_k \,\rrparenthesis) = 2$, we have to show that $x_i : \llparenthesis\, s_k \,\rrparenthesis \notin \omega^{+}(\Pi_{\mathbf{min}})$. Observe that **feed** and **feed_tag** either raise the nesting level of a state from 0 to 1, or leave it invariant if it was higher than 1. The instructions **tag**, **move** and **fuse** leave the nesting level unaltered, whereas **flush** reduces the nesting level to 0 or leaves it invariant. Therefore, the instruction set $I_{\mathbf{min}}$ does not contain any instruction that would increase the nesting level from 1 to 2. Therefore, $S \equiv x_i : \llparenthesis\, s_k \,\rrparenthesis$ is not constructable from $\emptyset$.

C.3.4. Subsubcase $P \equiv \llparenthesis\, P' + P'' \,\rrparenthesis$.
By induction, there exists a program π such that $\langle \pi, \emptyset \rangle \longrightarrow x_i : \llparenthesis\, P' \,\rrparenthesis \circ x_j : \llparenthesis\, P'' \,\rrparenthesis$. Then, for some $x_k \in \mathcal{L}_{\mathbf{feed\_tag}}$, $\sigma \in \mathcal{T}$, the program $\pi; \pi'$ will produce S, where π' is the following program:

$$
\begin{array}{ll}
 & x_i : \llparenthesis\, P' \,\rrparenthesis \circ x_j : \llparenthesis\, P'' \,\rrparenthesis \\
\mathbf{feed\_tag}(x_k, \sigma, 1) & x_i : \llparenthesis\, P' \,\rrparenthesis \circ x_j : \llparenthesis\, P'' \,\rrparenthesis \circ x_k : \sigma \\
\mathbf{move}(x_k, x_j, \sigma) & x_i : \llparenthesis\, P' \,\rrparenthesis \circ x_j : \llparenthesis\, P'' \,\rrparenthesis + \sigma \\
\mathbf{tag}(x_j) & x_i : \llparenthesis\, P' \,\rrparenthesis \circ x_j : \sigma \llparenthesis\, P'' \,\rrparenthesis \\
\mathbf{move}(x_j, x_i, \sigma) & x_i : \llparenthesis\, P' \,\rrparenthesis + \llparenthesis\, P'' \,\rrparenthesis \\
\mathbf{fuse}(x_i) & x_i : \sigma \llparenthesis\, P' + P'' \,\rrparenthesis \\
\mathbf{feed\_tag}(x_k, \sigma \triangleright \diamond, 1) & x_i : \sigma \llparenthesis\, P' + P'' \,\rrparenthesis \circ x_k : \sigma \triangleright \diamond \\
\mathbf{move}(x_k, x_i, \sigma \triangleright \diamond, 1) & x_i : \sigma \llparenthesis\, P' + P'' \,\rrparenthesis + \sigma \triangleright \diamond \\
 & x_i : \llparenthesis\, P' + P'' \,\rrparenthesis
\end{array}
$$

C.3.5. Subsubcase $P \equiv (s_k + q) \llparenthesis\, P' \,\rrparenthesis$.
By induction, there exists a program π such that $\langle \pi, \emptyset \rangle \longrightarrow x_i : q \llparenthesis\, P' \,\rrparenthesis$. Then, for any

$x_k \in \mathcal{L}_{\mathbf{feed\_tag}}$, the program $\pi; \pi'$ will produce S, where π' is the program:

$$\begin{array}{ll} & x_i : q\llparenthesis\ P\ \rrparenthesis \\ \mathbf{feed\_tag}(x_j, s_k, 1) & x_i : q\llparenthesis\ P\ \rrparenthesis \circ x_j : s_k \\ \mathbf{move}(x_j, x_i, s_k) & x_i : s_k + q\llparenthesis\ P\ \rrparenthesis \\ \mathbf{tag}(s_k) & x_i : (s_k + q)\llparenthesis\ P\ \rrparenthesis \end{array}$$

C.4. Subcase $P \equiv m_j$.
We have to show that $x_i : m_j \notin \omega^+(\Pi_{\mathbf{min}})$. First, observe that $d(x_i : m_j) = 0$. The only way to introduce m_j is by means of the command $\mathbf{feed}(x_i, m_j, 1)$ which will result in a nesting level of 1. As discussed in subcase C.3.3., the only means to decrease the nesting level is by means of the instruction **flush**. However, **flush** will transform the state $x_i : \llparenthesis\ m_j\ \rrparenthesis$ into the empty state, which is not equivalent to S. Thus, there is no program in $\Pi_{\mathbf{min}}$ able to generate $S \equiv x_i : m_j$. Therefore, $S \notin \omega^+(\Pi_{\mathbf{min}})$.

C.5. Subcase $P \equiv s_k$.
This is achieved by the program $\pi = \mathbf{feed\_tag}(x_j, s_k, 1); \mathbf{move}(x_j, x_i, s_k)$, for any $x_k \in \mathcal{L}_{\mathbf{feed\_tag}}$.

Proof of Theorem 4

Proof: The proof is identical to the one of theorem 3, where subsubcase C.3.3. is replaced by the more general subsubcase:

C.3.3' Subsubscase $P \equiv \llparenthesis\ P'\ \rrparenthesis$ with $d(P') > 0$, $P \not\equiv P' + P''$, and $P \not\equiv m_j$.
By induction, there exists a program $\pi \in \Pi_{\mathbf{wrap}}$, such that $\langle \pi, \emptyset \rangle \to x_j : P'$ for $x_j \in \mathcal{L}_{\mathbf{wrap\_in}}$. The desired state is obtained by the program $\pi; \pi'$, where π' is the program

$$\begin{array}{ll} & x_j : P' \\ \mathbf{wrap}(x_j, x_k) & x_k : \llparenthesis\ P'\ \rrparenthesis \\ \mathbf{transport}(x_k, x_i, x_l, \sigma) & x_i : \llparenthesis\ P'\ \rrparenthesis \end{array}$$

where $x_k \in \mathcal{L}_{\mathbf{wrap\_out}}$, $x_l \in \mathcal{L}_{\mathbf{feed\_tag}}$, $\sigma \in \mathcal{T}$.

Proof of Theorem 5

Proof: The proof is identical to the one of theorem 4 where case C.4. is replaced by:

C.4' Case $P \equiv m_j$.
By induction, there exists a program $\pi \in \Pi_{\mathbf{burst}}$, such that $\langle \pi, \emptyset \rangle \to x_i : \sigma\llparenthesis\ m_j\ \rrparenthesis$. The desired state is obtained by the program $\pi; \pi'$, where π' is the program

$$\begin{array}{ll} & x_i : \sigma\llparenthesis\ m_j\ \rrparenthesis \\ \mathbf{burst}(x_i) & x_i : \sigma\llparenthesis\ \ \rrparenthesis + m_j \\ \mathbf{move}(x_i, x_k, \sigma) & x_i : m_j \circ x_k : \sigma\llparenthesis\ \ \rrparenthesis \\ \mathbf{flush}(x_k) & x_i : m_j \end{array}$$

Proof of Theorem 6

Proof: Again, we proof the theorem by induction over the structure of S.

A. Case $S \equiv \emptyset$.
Nothing needs to be done in this case: $\langle \epsilon, \emptyset \rangle \to \emptyset$.

B. Case $S \equiv S' \circ S''$.
Just as in the proof for theorem 3, we know by induction that there exist π', π'' such that $\langle \pi', S' \rangle \to \emptyset$ and $\langle \pi'', S'' \rangle \to \emptyset$. If follows through lemmas 1 and 2 that $\langle \pi'; \pi'', S' \circ S'' \rangle \to \emptyset$.

C. Case $S \equiv x_i : P$.

C.1. Subcase $P \equiv 0$.
Nothing needs to be done in this case: $\langle \epsilon, \emptyset \rangle \to \emptyset$.

C.2. Subcase $P \equiv P' + P''$.
This is structurally equivalent to $x_i : P' \circ x_i : P''$ and is therefore reduced to case B.

C.3. Subcase $P \equiv q\llparenthesis\, P' \,\rrparenthesis$.

C.3.1. Subsubcase $q \equiv \diamond$.

	$x_i : \llparenthesis\, P \,\rrparenthesis$
feed_tag(x_j, σ)	$x_i : \llparenthesis\, P \,\rrparenthesis \circ x_j : \sigma$
move(x_j, x_i, σ)	$x_i : \sigma + \llparenthesis\, P \,\rrparenthesis$
tag(x_i)	$x_i : \sigma\llparenthesis\, P \,\rrparenthesis$

This reduces the problem to subsubcase C.3.2.

C.3.2. Subsubcase $q \equiv s_k + q'$.
For $x_j \in \mathcal{L}_{\mathbf{flush}}$, the following program resets the state:

	$x_i : (s_k + q)\llparenthesis\, P \,\rrparenthesis$
move(x_i, x_j, s_k)	$x_j : (s_k + q)\llparenthesis\, P \,\rrparenthesis$
flush(x_j)	$\emptyset$

C.4. Subcase $P \equiv m_j$.
If $x_i \in \mathcal{L}_{\mathbf{flush}}$, the program **flush**$(x_i)$ will reset the state. Likewise, if $x_i \in \mathcal{L}_{\mathbf{wrap\_in}}$, the program **wrap**$(x_i, x_j)$ will transform the state into $x_j : \llparenthesis\, m_j \,\rrparenthesis$ for any $x_j \in \mathcal{L}_{\mathbf{wrap\_out}}$ and reduces the problem to subcase C.3.1.. On the other hand, if $x_i \notin \mathcal{L}_{\mathbf{flush}} \cup \mathcal{L}_{\mathbf{wrap\_in}}$, we have to show that S is not an element of $\omega^-(\Pi_{\mathsf{x}})$. Note that all instructions **feed**, **feed_tag** **tag**, **move**, **burst** leave $x_i : m_j$ invariant under transition. Thus, there is no sequence of instructions that would transform $x_i : m_j$ into the empty state. Therefore, $x_i : m_j \notin \omega^-(\Pi_{\mathsf{x}})$ for $x_i \notin \mathcal{L}_{\mathbf{flush}} \cup \mathcal{L}_{\mathbf{wrap\_in}}$.

C.5. Subcase $P \equiv s_k$.
For $x_j \in \mathcal{L}_{\mathbf{flush}}$, the following program resets the state:

	$x_i : s_k$
move(x_i, x_j, s_k)	$x_j : s_k$
flush(x_j)	$\emptyset$